9.4 Fredholm Alternative for 2^{nd} order ODE. Of Sturm-Liouville type.

(I) $$\begin{cases} u'' + 4u = x, & 0 < x < 1 \\ u(0) = 0, \quad u(1) = 0. \end{cases}$$

Soln. for homog. BVP associated to this

$$u_h'' + 4u_h = 0, \quad u_h(0) = 0, \quad u_h(1) = 0.$$

$$\underline{r^2 + 4 = 0}$$

$$u_h(x) = C_1 \cos(2x) + C_2 \sin(2x)$$

A particular solution:

$$u_p(x) = A_0 + A_1 x$$

Then, $u_p' = A_1, \quad u_p'' = 0$

$$\Rightarrow \quad 4u_p = x \Rightarrow$$

$$4A_0 + 4A_1 x = x \Rightarrow A_0 = 0, \quad A_1 = \frac{1}{4}$$

$$\Rightarrow \quad u_p(x) = \frac{x}{4}$$

General Soln:

$$\boxed{u(x) = C_1 \cos(2x) + C_2 \sin(2x) + \frac{x}{4}}$$

Using BCs: $u(0) = 0$

$$\Rightarrow \boxed{0 = C_1} \Rightarrow u(x) = C_2 \sin(2x) + \frac{x}{4}$$

$$0 = u(1) \Rightarrow 0 = C_2 \sin(2) + \frac{1}{4}$$

$$\Rightarrow \boxed{C_2 = \frac{-1/4}{\sin(2)}}$$

Finally, the Soln. for the original problem is

$$\underbrace{\boxed{u(x) = -\frac{1}{4\sin(2)} \sin(2x) + \frac{x}{4}}}_{\text{unique solution}}$$

Associated EVP to (I):

$$\begin{cases} \phi''(x) + 4\phi(x) = \lambda \phi(x) \\ \phi(0) = 0, \quad \phi(1) = 0 \end{cases}$$

$$\phi(x) = C_1 \cos(\sqrt{4-\lambda}\, x) + C_2 \sin(\sqrt{4-\lambda}\, x)$$

Using BCs:

Assumig $4-\lambda > 0$

$$0 = \phi(0) = C_1 \Rightarrow \boxed{C_1 = 0}$$

$$0 = \phi(1) = C_2 \sin(\sqrt{4-\lambda})$$

For $C_2 \neq 0$

$$\Rightarrow \sqrt{4-\lambda} = n\pi \Rightarrow \boxed{\lambda_n = 4 - (n\pi)^2}$$

$n = 1, 2, 3, \ldots$

$$\boxed{\lambda_1 = 4 - \pi^2}, \quad \boxed{\lambda_2 = 4 - 4\pi^2}, \quad \boxed{\lambda_3 = 4 - 9\pi^2}, \ldots$$

So, $\lambda = 0$ is not an eigenvalue.

- The set of eigenvalues is $\lambda_n = 4 - (n\pi)^2$
- The corresponding set of eigenfunctions is $\phi_n(x) = \sin(n\pi x)$

$n = 1, 2, \ldots$

Using Green's formula and Green's function for (I)

$$L \equiv \frac{\partial^2}{\partial x^2} + 4$$

$$\int_0^1 \left[(Lu)G - (LG)u\right] dx =$$

$$= \int_0^1 \left[Lu\, G - LG\, u\right] dx = \left(u'\cancel{G}^{\,0} - G'\cancel{u}^{\,0}\right)\Big|_0^1 = 0$$

$$\Rightarrow \int_0^1 x\, G(x, x_0)\, dx - \int_0^1 u(x)\, \delta(x - x_0)\, dx = 0$$

$$\Rightarrow u(x_0) = \int_0^1 x\, G(x, x_0)\, dx$$

or $$u(x) = \int_0^1 x_0\, G(x, x_0)\, dx_0$$

or $$u(x) = \int_0^1 f(x_0)\, G(x, x_0)\, dx_0$$

From previous derivation

$$u(x) = \int_0^1 G(x,x_0)\, f(x_0)\, dx_0$$

$$= \int_0^1 \sum_{n=1}^{\infty} \frac{\phi_n(x)\, \phi_n(x_0)}{\int_0^1 \phi_n^2 dx \; \lambda_n} f(x_0)\, dx_0$$

$$= \int_0^1 \left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \frac{\sin(n\pi x)\, \sin(n\pi x_0)}{\int_0^1 \sin(n\pi)^2 x_0\, dx_0} \right) f(x_0)\, dx_0$$

(II) $\begin{cases} u'' + 4u = x, & 0 < x < \pi \\ u(0) = 0, \; u(\pi) = 0 \end{cases}$

Gen. Soln: $u(x) = C_1 \cos(2x) + C_2 \sin(2x) + \frac{x}{4}$

Using BCs:

$$0 = u(0) = C_1 \Rightarrow \boxed{C_1 = 0}$$

$$0 = u(\pi) = C_2 \underbrace{\sin(2\pi)}_{=0} + \frac{\pi}{4} = \frac{\pi}{4}$$

or $0 = \pi/4$?

implies no solution.

(III) $\begin{cases} u'' + 4u = \cos(2x), & 0 < x < \pi \\ u(0) = 0, \; u(\pi) = 0 \end{cases}$

Particular solution: Using the method of undetermined Coefficients.

$$u_p(x) = A x \cos(2x) + B x \sin(2x)$$

Substitution in governing equation leads to

$$\boxed{u_p(x) = \frac{x}{4} \sin(2x)}$$

Then general solution:

$$\boxed{u(x) = C_1 \cos(2x) + C_2 \sin(2x) + \frac{x}{4} \sin(2x)}$$

Using BCs:

$$0 = u(0) = C_1 \Rightarrow \boxed{C_1 = 0}$$

$$0 = u(\pi) = C_2 \overset{=0}{\sin(2\pi)} + \frac{\pi}{4} \overset{=0}{\sin(2\pi)} \checkmark$$

So, C_2 is undetermined

and the BVP has infinitely many solns.

$$\boxed{u(x) = C_2 \sin(2x) + \frac{x}{4} \sin(2x)}$$

The Eigenvalue Problem (EVP) associated to the original problem: (III) and (II)

$$\left\{ \boxed{\phi''(x) + 4\phi(x) = \lambda\phi, \; \phi(0) = 0, \; \phi(\pi) = 0} \right.$$

homog.

or $\phi''(x) + \overset{>0}{(4-\lambda)} \phi(x) = 0$ $\quad 4-\lambda > 0$ due to BCs.

Soln: $\phi(x) = c_1 \cos(\sqrt{4-\lambda}\, x) + c_2 \sin(\sqrt{4-\lambda}\, x)$

Using BCs:

$$0 = u(0) = c_1 \Rightarrow c_1 = 0.$$

$$0 = u(\pi) = C_2 \sin(\sqrt{4-\lambda}\,\pi)$$

To avoid Trivial soln.

$$\Rightarrow \sin(\sqrt{4-\lambda}\,\pi) = 0 \Rightarrow \sqrt{4-\lambda}\,\cancel{\pi} = n\cancel{\pi}$$

or $\boxed{\lambda_n = 4 - n^2}$, $n = 1, 2, \ldots$

$\lambda_1 = 3$, $\boxed{\lambda_2 = 0,}$ $\lambda_3 = -5, \ldots$

So $\boxed{\lambda_n = 4 - n^2}$

and $\boxed{\phi_n(x) = \sin(n\pi),}$ $n = 1, 2, \ldots$

With $\boxed{\phi_2(x) = \sin(2x)}$ the eigenfunction

Corresponding to $\boxed{\lambda_2 = 0.}$

Notice that

$$\int_0^{\pi} \sin(2x)\cos(2x)\,dx = \frac{1}{2}\int_0^{\pi} \sin(4x)\,dx$$

$$= -\frac{1}{2}\,\frac{\cos(4x)}{4}\Big|_0^{\pi} = \frac{-1}{2}\left(\frac{1}{4} - \frac{1}{4}\right) = 0$$

or

Case III $\displaystyle\int_0^{\pi} \overset{\lambda_2=0}{\phi_2(x)}\, \underset{\cos(2x)}{\underset{\parallel}{f(x)}}\,dx = 0 \Rightarrow$ Infinitely many Solutions

Also,

Case II $\displaystyle\int_0^{\pi} \underset{f}{x}\,\underset{g'}{\sin(2x)}\,dx = -\frac{x}{2}\cos(2x)\Big|_0^{\pi} - \int_0^{\pi} \sin 2x\,dx$

$$= -\frac{\pi}{2} + \frac{\cos(2x)}{2}\Big|_0^{\pi} = -\frac{\pi}{2} + 0 = -\pi/2$$

$$\neq 0.$$

$\Rightarrow$ No Solutions.

Fredholm Alternative for both

<u>BVP. for ODE's</u> (Sturm - Liouville form)

$$\boxed{\begin{cases} L\,u(x) = f(x), & 0 < x < \\ u(a) = 0, \quad u(b) = 0 \end{cases}} \quad \begin{matrix} (1.1) \\ (1.2) \end{matrix}$$

and <u>Algebraic Linear System</u> of equations.

$$\boxed{A\vec{x} = \vec{b}.} \qquad (1.3)$$

<u>Two Cases:</u>

i) $\lambda = 0$ is not an eigenvalue of neither (1.1)-(1.2) and (1.3).

Corresponding eigenvalue problems are respectively

$$\boxed{\begin{cases} L\,\phi(x) = \lambda\phi(x), & 0 < x < L \\ \phi(0) = 0, \quad \phi(L) = 0. \end{cases}} \quad \begin{matrix} (1.4) \\ (1.5) \end{matrix}$$

and

$$\boxed{A\vec{x} = \lambda\vec{x}} \qquad (1.6)$$

If L is a S-L differential operator then the eigenfunctions $\phi_n(x)$ corresponding to λ_n form a complete set and they are orthogonal, then ?

$$\boxed{u(x) = \sum_{n=1}^{\infty} u_n \phi_n(x)} \qquad (2.1)$$

and $$\boxed{f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x),} \qquad f_n = \int_0^L f(x)\phi_n(x)\,dx \qquad (2.2)$$

We will choose an orthonormal set $\{\phi_n(x)\}$.

As a result, (1.1) can be written as

$$L\left(\sum_n u_n \phi_n\right) = \sum_n f_n \phi_n(x)$$

or $$\sum_{n=1}^{\infty} u_n L\phi_n = \sum f_n \phi_n$$

or $$\sum_{n=1}^{\infty} u_n \lambda_n \phi_n = \sum_{n=1}^{\infty} f_n \phi_n$$

Using orthogonality

$$u_n \lambda_n = f_n \xRightarrow{\lambda_n \neq 0} \boxed{u_n = \frac{f_n}{\lambda_n}} \qquad (2.3)$$

is uniquely determined

and

$$u(x) = \sum_n u_n \phi_n = \sum_n \frac{f_n}{\lambda_n} \phi_n$$

$$= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left(\int_0^L f(x_0) \phi_n(x_0)\, dx_0 \right) \phi_n(x)$$

$$= \boxed{\int_0^L \left(\sum_{n=1}' \frac{\phi_n(x)\, \phi_n(x_0)}{\lambda_n} \right) f(x_0)\, dx_0} \quad (3.1)$$

Eigenfunction expansion Green's function

$$= \boxed{\int_0^L G(x, x_0)\, f(x_0)\, dx_0} \quad (3.2)$$

In summary, if $\lambda_n \neq 0$, for all n, then (1.1) has a unique solution given by (3.1)-(3.2).

Similarly, if $\lambda = 0$ is not an eigenvalue of (1.3), then $A_{n\times n}$ is invertible, which is equivalent to say that the columns of A form a basis for $\mathbb{R}^n$. Using Grand-Schmidt we can obtain an orthonormal basis B from the columns of A. Moreover, if A is symmetric B is formed by orthonormal eigenvectors.

It means

$$B = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \}$$

is such that $\quad V_i \cdot V_j = 0 \quad \|V_i\| = 1$

and

$$\boxed{A V_i = \lambda_i V_i.} \qquad (4.1)$$

Now, we can obtain a soln. of (1.3) with respect to the base B, as follows:

$$A\vec{x} = \vec{b}, \qquad \boxed{\begin{aligned} \vec{x} &= c_1\vec{V}_1 + c_2\vec{V}_2 + \cdots + c_n\vec{V}_n \quad (4.2) \\ \vec{b} &= b_1\vec{V}_1 + b_2\vec{V}_2 + \cdots + b_n\vec{V}_n \end{aligned}}$$

Can be expressed as

$$A\left(c_1\vec{v}_1 + \overset{+c_j\vec{v}_j}{\cdots} + c_n\vec{v}_n\right) = b_1\vec{v}_1 + \cdots + b_n\vec{v}_n$$

$$\Rightarrow \quad c_1 A\vec{v}_1 + \overset{+c_jA\vec{v}_j}{\cdots} + c_n A\vec{v}_n) = \sum_{i=1}^{n} b_i V_i$$

or

$$c_1\lambda_1\vec{v}_1 + \overset{+c_j\lambda_j\vec{v}_j}{\cdots} + c_n\lambda_n\vec{v}_n = \sum_{i=1}^{n} b_i\vec{v}_i$$

Using orthogonality

$$c_j\lambda_j = b_j \;\overset{\lambda_j \neq 0}{\Rightarrow}\; \boxed{c_j = \frac{b_j}{\lambda_j}} \quad (4.3)$$

are uniquely determined

and

$$\boxed{\vec{x} = \sum_{i=1}^{n}{}' \frac{b_i}{\lambda_i} V_i} \quad (4.4)$$

Case 2

ii) What if $\lambda_j = 0$

then,

$$L\left(\sum_n u_n \phi_n\right) = \sum_n f_n \phi_n$$

or

$$\sum_n u_n \underbrace{L\phi_n}_{\lambda_n \phi_n} = \sum f_n \phi_n$$

Orthog. $\Rightarrow$

$$u_n \lambda_n = f_n, \qquad n = 1, 2 \ldots j, \ldots n$$

In particular, at $n = j$

$$u_j \overset{=0}{\lambda_j} = f_j \Rightarrow 0 = f_j$$

Therefore for a soln to exist $\boxed{f_j = 0}$ and u_j arbitrary

but $f_j = \int_0^L f(x)\, \phi_j(x)\, dx. = 0$

which implies $f \perp \phi_j$, for a solution to exist.*

iii) Otherwise, if $\lambda_j = 0$ and $f \not\perp \phi_j$ (f not orthogonal to ϕ_j) then there is no solution.

* Actually in this case, there are infinitely many solutions.

Case 2: For Algebraic Linear Systems.

ii) If $\lambda_j = 0$

then $c_j \overset{=0}{\lambda_j} = b_j \Rightarrow 0 = b_j$

Therefore, the only possibility for solution is if $b_j = 0$. This is equivalent to

$$\boxed{\vec{b} \cdot \vec{V}_j = 0} \qquad \vec{b} \perp \vec{V}_j$$

Since

$$\Leftrightarrow \left(b_1\vec{v}_1 + b_2\vec{v}_2 + \dots + b_j\vec{v}_j + \dots + b_n\vec{v}_n\right)\cdot\vec{V}_j = 0$$

$$\Leftrightarrow \quad b_j\underbrace{(\vec{V}_j \cdot \vec{V}_j)}_{=1} = 0 \Leftrightarrow b_j = 0$$

iii) Otherwise, $\vec{b} \not\perp \vec{V}_j$ and there is no solution.

Green's formula for Linear Algebraic Systems

Consider $\vec{v}, \vec{u}$ in $\mathbb{R}^n$ and $A_{n\times n}$.

Then,

$$\vec{u} \cdot A\vec{v} - \vec{v} \cdot A\vec{u}$$

$$= \sum_i \left(u_i \sum_j a_{ij} v_j \right) - \sum_i \left(v_i \sum_j a_{ij} u_j \right)$$

$$= \sum_i u_i \sum_j a_{ij} v_j - \sum_j v_j \sum_i a_{ji} u_i \qquad (5.1)$$

$$= 0, \quad \text{if} \quad a_{ij} = a_{ji} \text{ or } A^T = A$$

Symmetric matrix

What is that formula good for?

Prove that eigenvectors $\vec{v}_1$ and $\vec{v}_2$ corresponding to different eigenvalues λ_1 and λ_2, $\lambda_1 \neq \lambda_2$ are orthogonal.

$$A\vec{v}_1 = \lambda_1 \vec{v}_1, \quad A\vec{v}_2 = \lambda_2 \vec{v}_2$$

Applying Green's form for Alg. syst

$$\vec{v}_1 \cdot A\vec{v}_2 - \vec{v}_2 \cdot A\vec{v}_1 = 0, \quad \text{if } A = A^T$$

Then $v_1 \cdot \lambda_2 v_2 - v_2 \cdot \lambda_1 v_1 = 0 \Leftrightarrow \overset{\neq 0}{(\lambda_2 - \lambda_1)} \vec{v}_1 \cdot \vec{v}_2 = 0$

$$\Rightarrow \vec{v}_1 \cdot \vec{v}_2 = 0$$